Solution. Let $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$. Here $a_{ij}\in\mathbb{R};\ i,j=1,2,...,n$.

We need to show $\left(A+A^T\right)^T=A+A^T$

$$A+A^T=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}+\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{bmatrix}=\begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \cdots & a_{1n}+a_{n1} \\ a_{21}+a_{12} & a_{22}+a_{22} & \cdots & a_{2n}+a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+a_{1n} & a_{n2}+a_{2n} & \cdots & a_{nn}+a_{nn} \end{bmatrix}$$

$$\underset{a,b\in\mathbb{R}\Rightarrow a+b=b+a}{=}\begin{bmatrix} a_{11}+a_{11} & a_{21}+a_{12} & \cdots & a_{n1}+a_{1n} \\ a_{12}+a_{21} & a_{22}+a_{22} & \cdots & a_{n2}+a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+a_{n1} & a_{2n}+a_{n2} & \cdots & a_{nn}+a_{nn} \end{bmatrix}=\left(A+A^T\right)^T \Rightarrow A+A^T \text{ is symmetric matrix.}$$

Second Solution. Or we can use properties of matrices.

Let A be a square matrix.

$$\left(A+A^T\right)^T \underset{(A+B)^T=A^T+B^T}{=} A^T+\left(A^T\right)^T \underset{\left(A^T\right)^T=A}{=} A^T+A \underset{A+B=B+A}{=} A+A^T$$

2. Let $A=\begin{bmatrix} 3 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find inverse of A, if it exists.

$$A^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 1 & -3 & -2 \\ -1 & 3 & 3 \end{bmatrix}$$

3. Let A, B, C and D be $3\times4-$matrices such that

$$A \xrightarrow{-R_1+R_3} B \xrightarrow{R_1 \leftrightarrow R_2} D \quad \text{and} \quad C \xrightarrow{3R_2+R_3} D.$$

Find an invertible matrix P such that $PA = C$ and write P as a product of 3 elementary matrices accordingly to the diagrams above **(20p).**

Solution. $A \xrightarrow{\varepsilon_1 : -R_1+R_3} B \xrightarrow{\varepsilon_2 : R_1 \leftrightarrow R_2} D \xrightarrow{\varepsilon_3 : -3R_2+R_3} C$

$$P = \varepsilon_3(I).\varepsilon_2(I).\varepsilon_1(I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

4. $$\begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + \left(k^2 - 5\right) z = k \end{cases}$$

For the linear system, which is given above, determine all values of k for which the resulting linear system has

a) no solution;

b) a unique solution;

c) infinitely many solutions **(20p).**

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & 2 & 1 & 3 \\ 1 & 1 & k^2-5 & k \end{array}\right] \xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & k^2-4 & k-2 \end{array}\right]$$

a) For $k=-2$ the system has no solutions.

b) For $k \neq \pm 2$ the system has unique solutions.

c) For $k=2$ the system has infinitely many solutions.

5. Find the fundamental solutions and general solution of the following system

$$\begin{cases} 2x+2y-z+u=0 \\ -x-y+2z-3t+u=0 \\ x+y-2z-u=0 \\ x+y+u=0 \end{cases} \quad \textbf{(20p).}$$

Solution.

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_2+R_4]{\substack{2R_2+R_1 \\ R_2+R_3}} \begin{bmatrix} 0 & 0 & 3 & -6 & 3 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 2 & -3 & 2 \end{bmatrix} \xrightarrow[-R_3/3]{\substack{R_1/3 \\ -R_2}} \begin{bmatrix} 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow[3R_3+R_4]{\substack{2R_3+R_1 \\ -3R_3+R_2}} \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 2 \end{bmatrix} \xrightarrow[R_4/2]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{-R_2+R_4} \begin{bmatrix} \boxed{1} & 1 & -2 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

y and u are free variables.

F1) Let $y=1, u=0 \Rightarrow \begin{cases} x+1-2z=0 \Rightarrow x=-1 \\ z=0 \\ t=0 \end{cases} \Rightarrow X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$F2)$ Let $y=0, u=1 \Rightarrow \begin{cases} x-2z-1=0 \Rightarrow x=-1 \\ z+1=0 \Rightarrow z=-1 \\ t=0 \end{cases} \Rightarrow X_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

X_1 and X_2 are fundamental solutions of the system. General solution is $X = yX_1 + uX_2$.

Exercise. Find the values of k for which the matrix equation

$$x\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+y\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}+z\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}+t\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}=\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}$$

has a solution, and find, for these k, the general solution to the given equation.

Solution. $\begin{bmatrix}x & y+t\\ y+z & z-t\end{bmatrix}=\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}\Rightarrow\begin{cases}x=1\\ y+t=k\\ y+z=0\\ z-t=1\end{cases}$

$$\left[A|B\right]=\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & k\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & -1 & 1\end{array}\right]\xrightarrow[-R_3+R_4]{-R_2+R_3}\left[\begin{array}{cccc|c}\boxed{1} & 0 & 0 & 0 & 1\\ 0 & \boxed{1} & 0 & 1 & k\\ 0 & 0 & \boxed{1} & -1 & -k\\ 0 & 0 & 0 & 0 & k+1\end{array}\right].$$

t is free variable. If the solution exists, $k=-1$.

$$\left[\begin{array}{cccc|c}1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 1 & -1\\ 0 & 0 & 1 & -1 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

$$t=1\Rightarrow\begin{cases}x=0\\ y+1=0\Rightarrow y=-1, z=1\Rightarrow X_1=\begin{bmatrix}0\\ -1\\ 1\\ 1\end{bmatrix}\\ z-t=0\end{cases}$$

Hence the general solution to the homogeneous system is $X_g=tX_1=t\begin{bmatrix}0\\ -1\\ 1\\ 1\end{bmatrix}$

To find the particular solution as $t=0$

$$\begin{cases}x=1\\ y=-1\Rightarrow y=-1, z=1\Rightarrow X_p=\begin{bmatrix}1\\ -1\\ 1\\ 0\end{bmatrix}\\ z=1\end{cases}$$

Therefore, the general solution of the system is

$$X=X_g+X_p=t\begin{bmatrix}0\\ -1\\ 1\\ 1\end{bmatrix}+\begin{bmatrix}1\\ -1\\ 1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ -t-1\\ t+1\\ t\end{bmatrix}$$

ADDITIONAL EXERCISES

1. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix}.$$

Find A^{-1} (the inverse of A) if it exists.

Answer: $A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ -1/2 & 1 & -1/2 \\ 5/2 & -2 & 1/2 \end{bmatrix}$

2. Let

$$A = \begin{bmatrix} 3 & -3 & 7 & 2 \\ 1 & -1 & 3 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

a) Find a row reduced echelon matrix R that is row equivalent to A.

b) Find the value(s) of k (if exist) for which A is row equivalent to B.

Answer: a) $R = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 0 & k & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$ and b) $k = 1$.

3. Let C, D, L, M and K be $2\times 4-$matrices such that

$$C \xrightarrow{R_1 \leftrightarrow R_2} L \xrightarrow{2R_2} K \text{ and } D \xrightarrow{2R_2+R_1} M \xrightarrow{3R_1} K$$

Find an invertible matrix P such that $PC = D$ and write P as a product of 4 elementary matrices accordingly to the diagrams above.

Answer: $P = \begin{bmatrix} -4 & 1/3 \\ 2 & 0 \end{bmatrix}$

4. Compute $A^2 - A + 3I$ for

$$A = \begin{bmatrix} -1 & -3 & 1 \\ 0 & -2 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

5. Determine all matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ satisfying $A^2 - I = O$.

6. Find a row-reduced echelon matrix R which is row equivalent to

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & -2 & -2 & -1 \\ 2 & 0 & 1 & 7 & 2 \end{bmatrix}$$

7. Determine whether

$$\begin{bmatrix} -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

is invertible or not. If it is invertible, find its inverse.

8. Find the general solution of the system

$$\begin{cases} x - 2y - z + 2t = 0 \\ -x + 2y + 2z - t = 2 \\ x + y + z + t = 4 \\ x + y + 2z + 2t = 6 \end{cases}$$

and write four distinct solutions.

9. Find the general solution of the homogeneous system

$$\begin{cases} x - y - z - 2t = 0 \\ x + y + 2z + 3t = 0 \\ -x + 2y + z - t = 0 \\ x + 2y + 2z = 0 \end{cases}$$

10. a) Find the value(s) of r such that the following system of linear equations

$$\begin{cases} 2x + 3y + 7z + 11t = 1 \\ x + 2y + 4z + 7t = 2r \\ 5x + 10z + 5t = r - 1 \end{cases}$$

is consistent.

$$\begin{cases} 2x+3y+7z+11t=0 \\ x+2y+4z+7t=0. \\ 5x+10z+5t=0 \end{cases}$$

Answer: a) $r=\dfrac{11}{31}$ b) $X=z\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\-3\\0\\1\end{bmatrix}$.

11. Let $A=\begin{bmatrix}0&1&-1&1\\1&0&-2&3\\0&1&0&1\end{bmatrix}$ and $X=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$. Consider the homogeneous system $AX=0$.

Find for the system:

a) Free variable(s) and basic variable(s).

b) Fundamental solution(s).

c) The general solution.

d) Is the system $AX=B$ consistent for $B=\begin{bmatrix}2\\1\\0\end{bmatrix}$?

Answer b) $X=x_4\begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}$ **d)** The system is consistent (you need to show why).

Find the value(s) of t for which the following matrix equation has no solution

$$x\begin{bmatrix}1&2\\1&4\end{bmatrix}+y\begin{bmatrix}1&2\\0&3\end{bmatrix}+z\begin{bmatrix}1&3\\1&5\end{bmatrix}=\begin{bmatrix}2&5\\-3&t\end{bmatrix}.$$

Answer The equation has no solution iff $t\neq 4$.

12. Find the conditions on $a,\ b,\ c,$ and d for which the matrix system

$$x_1\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+x_2\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}+x_3\begin{bmatrix}3 & 3\\ 1 & 1\end{bmatrix}+x_4\begin{bmatrix}7 & 7\\ -3 & -3\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

has

a) no solution;

b) infinitely many solutions.

c) Find the general solution of the equation for $\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$.

16. Find the general solution of the system

$$\begin{cases} 2x-y-z=0 \\ 2x-y+4z=-1 \\ -x+2y+z=2 \end{cases}.$$

17. Find x, y, and z if

$$x\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+y\begin{bmatrix}1\\-1\\1\\1\end{bmatrix}+z\begin{bmatrix}1\\1\\3\\-1\end{bmatrix}=\begin{bmatrix}1\\0\\2\\5\end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 3 & 2 \\ 1 & 1 & -1 & 5 \end{array}\right] \xrightarrow[-R_1+R_4]{\substack{-R_1+R_2 \\ -3R_1+R_3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & -2 & 4 \end{array}\right] \xrightarrow[-\frac{1}{2}R_4]{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right] \xrightarrow[R_3\leftrightarrow R_4]{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & 0 & 1/2 \\ 0 & 0 & \boxed{1} & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x+y+z=1 \\ y=1/2 \\ z=-2 \end{array}\right\} \Rightarrow x=1-y-z=\frac{5}{2} \Rightarrow \text{ So the general solution is } X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5/2\\1/2\\-2\end{bmatrix}.$$

$$\begin{cases} x+y-z+t+u=1 \\ -x+2y+3z-t+2u=-1 \\ 2x+y-z+2t-u=2 \\ x+6y+4z+t+4u=1 \\ 8y+7z+6u=0 \\ 3x+7y+3z+3t+3u=3 \end{cases}.$$

Solution.

$$\left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ -1 & 2 & 3 & -1 & 2 & -1 \\ 2 & 1 & -1 & 2 & -1 & 2 \\ 1 & 6 & 4 & 1 & 4 & 1 \\ 0 & 8 & 7 & 0 & 6 & 0 \\ 3 & 7 & 3 & 3 & 3 & 3 \end{array}\right] \xrightarrow[\substack{-R_1+R_4 \\ -3R_1+R_6}]{\substack{R_1+R_2 \\ -2R_1+R_3}} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 0 & 3 & 0 \\ 0 & -1 & 1 & 0 & -3 & 0 \\ 0 & 5 & 5 & 0 & 3 & 0 \\ 0 & 8 & 7 & 0 & 6 & 0 \\ 0 & 4 & 6 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{8R_3+R_5 \\ 4R_3+R_6}]{\substack{3R_3+R_2 \\ 5R_3+R_4}} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 5 & 0 & -6 & 0 \\ 0 & -1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 10 & 0 & -12 & 0 \\ 0 & 0 & 15 & 0 & -18 & 0 \\ 0 & 0 & 10 & 0 & -12 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{\frac{1}{3}R_5 \\ \frac{1}{2}R_6}]{\substack{R_2+R_3 \\ \frac{1}{2}R_4}} \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & -3 & 0 \\ 0 & 0 & 5 & 0 & -6 & 0 \\ 0 & 0 & 5 & 0 & -6 & 0 \\ 0 & 0 & 5 & 0 & -6 & 0 \\ 0 & 0 & 5 & 0 & -6 & 0 \end{array}\right] \xrightarrow[-R_3+R_6]{\substack{-R_3+R_4 \\ -R_3+R_5}} \left[\begin{array}{ccccc|c} \boxed{1} & 1 & -1 & 1 & 1 & 1 \\ 0 & \boxed{-1} & 1 & 0 & -3 & 0 \\ 0 & 0 & \boxed{5} & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

t and u are free variables.

$$\left.\begin{array}{l} x+y-z+t+u=1 \\ -y+z-3u=0 \\ 5z-6u=0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} z=6u/5 \\ y=z-3u=\dfrac{6u}{5}-3u=-\dfrac{9u}{5} \\ x=1-y+z-t-u=1+\dfrac{9u}{5}+\dfrac{6u}{5}-t-u=1-t+2u \end{array}\right.$$

Hence the general solution of the given system is

$$X=\begin{bmatrix} x \\ y \\ z \\ t \\ u \end{bmatrix}=\begin{bmatrix} 1-t+2u \\ -9u/5 \\ 6u/5 \\ t \\ u \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}+u\begin{bmatrix} 2 \\ -9/5 \\ 6/5 \\ 0 \\ 1 \end{bmatrix}.$$